Ain Shams Engineering Journal (2015) 6, 1121–1127

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Numerical treatment of singular perturbation problems exhibiting dual boundary layers

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Received 1 December 2014; revised 30 January 2015; accepted 22 February 2015
Available online 29 March 2015

KEYWORDS

Singular perturbation problem;
Dual layer;
Fitting factor;
Tridiagonal system;
Discrete invariant imbedding algorithm

Abstract In this paper, we employed a fitted operator finite difference method on a uniform mesh for solving singularly perturbed two-point boundary value problems exhibiting dual boundary layers. In this method, we have extended the Numerov method to the second order singularly perturbed two-point boundary value problem with first order derivative. By using nonsymmetric finite differences for the first order derivative, we have derived the finite difference scheme. A fitting factor is introduced in this finite difference scheme which takes care of the rapid changes that occur in the boundary layer. This fitting factor is obtained from the asymptotic approximate solution of singular perturbations. Discrete invariant imbedding algorithm is used to solve the tridiagonal system of the fitted finite difference method. We have discussed the convergence analysis of the proposed method. Maximum absolute errors of the several numerical examples are presented to illustrate the proposed method.

1. Introduction

Singular perturbation problem now is a maturing mathematical subject with fairly long history and a strong promise for continued important applications throughout science and engineering. A singular perturbation problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the interval, as the perturbation parameter $\varepsilon \to 0$. Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution derivatives are extremely large. The numerical treatment of singularly perturbed differential equations gives major computational difficulties due to the presence of boundary and/or interior layers. If we apply the existing standard numerical methods for solving these problems, large

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2015.02.012
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oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior. Thus more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems.

The survey papers by Kadalbajoo and Reddy [1], Kadalbajoo and Patidar [2] give an erudite outline of the singular perturbation problems and their treatment on fluid dynamical boundary layers. These survey articles will remain as the most readable sources on singular perturbations. Abrahamsson [3] derived a priori estimates for the solutions of SPPs with a turning point. A set of general sufficient conditions for a uniformly convergent scheme for singularly perturbed turning point problem is obtained by Farrell [4]. Natesan and Ramanujam [5] derived a computational method for the singularly perturbed turning point problem in which exponentially fitted difference schemes are combined with classical numerical methods. Another technique known as initial-value technique was extended in [6] for the singularly perturbed turning point problem in which the numerical solution is obtained by solving suitable initial and terminal value problems. Natesan et al. [7] proposed a parameter uniform numerical method on Shishkin mesh to solve singularly perturbed turning point problems. Miller et al. [8] elucidate the classical schemes on Shishkin meshes to solve singularly perturbed BVPs of convection – diffusion and reaction – diffusion problems subject to Dirichlet boundary conditions.

In this paper, we employed a fitted operator finite difference method on a uniform mesh for solving singularly perturbed two-point boundary value problems exhibiting dual boundary layers. In Section 2, we described the fitted finite difference method by extending the Numerov method to the second order singularly perturbed two-point boundary value problem with first order derivative. In Section 3, we discussed the convergence analysis of the proposed method. To demonstrate the efficiency of the proposed method, numerical experiments are carried out for several test problems and the results are given in Section 4. Finally the discussions and conclusion are given in the last section.

2. Description of the method

Consider singularly perturbed boundary value problems of the form:

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), -1 \leqslant x \leqslant 1, \tag{1}$$

with boundary conditions $y(-1) = \alpha$ (2a)

and $y(1) = \beta$ (2b)

where $0 < \varepsilon \ll 1$, α and β are finite constants.

Here, we assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions such that

$$\begin{aligned} &a(0) = 0, && a'(0) \le 0, \\ &|a(x)| \ge a_0 > 0, && \text{for } 0 < x \le 1, \\ &b(x) \geqslant b_0 > 0, && \forall x \in D = [-1, 1], \\ &|a'(x)| \ge \tfrac{|a'(0)|}{2}, && \forall x \in D. \end{aligned}$$

With the above assumption, the turning point problem (1)–(2) possesses unique solution exhibiting two boundary layers of exponential type at both end points $x = -1, 1$.

Divide the interval $[-1, 1]$ into N equal parts with mesh size h, i.e., $h = \frac{2}{N}$ and $x_i = -1 + ih$ for $i = 0, 1, \ldots, N$. Let us denote $\frac{N}{2} = l$. Then, divide the interval $[-1, 1]$ into two subintervals $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, l-1$; and $[x_i, x_{i+1}]$ for $i = l+1, l+2, \ldots, N-1$. For dual layer problem, in the interval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, l-1$ layer exists at left end point and in $[x_i, x_{i+1}]$ for $i = l+1, l+2, \ldots, N-1$ layer is at right end point. Hence, we derive the numerical method for both left-end layer in $[-1, 0]$ and right-end layer in $[0, 1]$ cases.

In the interval $[-1, 0]$, from the theory of singular perturbation is well known that zeroth order asymptotic approximation to the solution of Eq. (1) is (cf. O'Malley [9])

$$y_i \approx y_0(-1 + ih) + (\alpha - y_0(-1)) \exp\left\{-\left(\frac{a(-1)}{\varepsilon}\right)(-1 + ih)\right\}.$$

Therefore

$$\lim_{h\to 0} y_i \approx y_0(-1) + (\alpha - y_0(-1)) \exp\left(-a(-1)\left(\frac{-1}{\varepsilon} + i\rho\right)\right) \tag{3}$$

where $\rho = \frac{h}{\varepsilon}$.

By the Numerov method, we have

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h^2}{12}(y''_{i-1} + 10y''_i + y''_{i+1}) + O(h^6) \tag{4}$$

Now, we extend this method for second order singular perturbation boundary value problem with first order derivative as follows:

From the Eq. (1), we have

$$\varepsilon y''_{i+1} = -a_{i+1}y'^*_{i+1} - b_{i+1}y_{i+1} + f_{i+1} \tag{5a}$$

$$\varepsilon y''_i = -a_i y'_i - b_i y_i + f_i \tag{5b}$$

$$\varepsilon y''_{i-1} = -a_{i-1}y'^*_{i-1} - b_{i-1}y_{i-1} + f_{i-1} \tag{5c}$$

We approximate y'^*_{i+1}, y'^*_{i-1} using nonsymmetric finite differences and y'_i by upwind finite difference

$$y'^*_{i+1} = \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} + O(h^2) \tag{6a}$$

$$y'_i = \frac{y_{i+1} - y_i}{h} + O(h) \tag{6b}$$

$$y'^*_{i-1} = \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} + O(h^2) \tag{6c}$$

Substituting Eqs. (5) and (6) in Eq. (4) and simplifying, we get

$$\begin{aligned} &\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}(-3y_{i-1} + 4y_i - y_{i+1}) \\ &\quad + \frac{10a_i}{12h}(y_{i+1} - y_i) + \frac{a_{i+1}}{24h}(y_{i-1} - 4y_i + 3y_{i+1}) + \frac{b_{i-1}}{12}y_{i-1} \\ &\quad + \frac{10b_i}{12}y_i + \frac{b_{i+1}}{12}y_{i+1} = \frac{(f_{i-1} + 10f_i + f_{i+1})}{12} \end{aligned}$$

Now, introducing the fitting factor $\sigma(\rho)$ (also called artificial viscosity) in the above scheme

$$\sigma(\rho)\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)+\frac{a_{i-1}}{24h}(-3y_{i-1}+4y_i-y_{i+1})$$
$$+\frac{10a_i}{12h}(y_{i+1}-y_i)+\frac{a_{i+1}}{24h}(y_{i-1}-4y_i+3y_{i+1})+\frac{b_{i-1}}{12}y_{i-1}$$
$$+\frac{10b_i}{12}y_i+\frac{b_{i+1}}{12}y_{i+1}=\frac{(f_{i-1}+10f_i+f_{i+1})}{12} \quad (7)$$

The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of (7) converges uniformly to the solution of Eqs. (1)-(2).

By using Eq. (3), we get

$$\underset{h\to 0}{Lt}(y_{i-1}-2y_i+y_{i+1})=(\alpha-y_0(-1))e^{-a(-1)(\frac{-1}{\varepsilon}+i\rho)}(e^{a(-1)\rho}+e^{-a(-1)\rho}-2) \quad (8a)$$

$$\underset{h\to 0}{Lt}(-3y_{i-1}+4y_i-y_{i+1})=(\alpha-y_0(-1))e^{-a(-1)(\frac{-1}{\varepsilon}+i\rho)}(-3e^{a(-1)\rho}-e^{-a(-1)\rho}+4) \quad (8b)$$

$$\underset{h\to 0}{Lt}(y_{i-1}-4y_i+3y_{i+1})=(\alpha-y_0(-1))e^{-a(-1)(\frac{-1}{\varepsilon}+i\rho)}(e^{a(-1)\rho}+3^{-a(-1)\rho}-4) \quad (8c)$$

$$\underset{h\to 0}{Lt}(y_{i+1}-y_i)=(\alpha-y_0(-1))e^{-a(-1)(\frac{-1}{\varepsilon}+i\rho)}(e^{-a(-1)\rho}-1) \quad (8d)$$

Multiplying Eq. (7) by h and taking the limit as $h\to 0$ (cf. Doolan et. al. [10]), and then using Eq. (8), we get

$$\sigma=\rho\frac{a(-1)}{12}\left(\frac{(1+12(1-\exp(-a(-1)\rho)))}{4\sinh^2\left(\frac{a(-1)\rho}{2}\right)}\right) \quad (9)$$

is a fitting factor in the interval $[-1, 0]$.

The tridiagonal system of the Eq. (7) can be written as follows:

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i, \quad i=1,2,\ldots,l-1 \quad (10)$$

where

$$E_j=\frac{\varepsilon\sigma}{h^2}-\frac{3a_{i-1}}{24h}+\frac{b_{i-1}}{12}+\frac{a_{i+1}}{24h}$$
$$F_j=\frac{2\varepsilon\sigma}{h^2}-\frac{4a_{i-1}}{24h}-\frac{10b_i}{12}+\frac{4a_{i+1}}{24h}+\frac{10a_i}{12h}$$
$$G_j=\frac{\varepsilon\sigma}{h^2}-\frac{a_{i-1}}{24h}+\frac{b_{i+1}}{12}+\frac{10a_i}{12h}+\frac{3a_{i+1}}{24h}$$
$$H_j=\frac{1}{12}(f_{i-1}+10f_i+f_{i+1})$$

where σ is given by Eq. (9).

Finally, we discuss our method in the interval [0, 1] for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval.

In the interval [0, 1], from the theory of singular perturbation the zeroth order asymptotic approximation to the solution of Eq. (1) is

$$y_i\approx y_0(-1+ih)+(\beta-y_0(1))\exp\left\{\left(\frac{a(1)}{\varepsilon}\right)(1-ih)\right\}$$

$$\lim_{h\to 0} y(ih)\approx y_0(-1)+\left(\beta-y_0(1)\exp\left\{a(1)\left(\frac{1}{\varepsilon}-i\rho\right)\right\}\right)$$
$$\text{where } \rho=\frac{h}{\varepsilon} \quad (11)$$

Now, for the right end boundary layer, we approximate $y'^{*}_{i+1}, y'^{*}_{i-1}$ using nonsymmetric finite differences and y'_i by backward finite difference

$$y'^{*}_{i+1}\approx\frac{y_{i-1}-4y_i+3y_{i+1}}{2h}+O(h^2) \quad (12a)$$

$$y'_i\approx\frac{y_i-y_{i-1}}{h}+O(h) \quad (12b)$$

$$y'^{*}_{i-1}\approx\frac{-3y_{i-1}+4y_i-y_{i+1}}{2h}+O(h^2) \quad (12c)$$

Substituting Eqs. (5) and (12) in Eq. (4) and simplifying we get

$$\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)+\frac{a_{i-1}}{24h}(-3y_{i-1}+4y_i-y_{i+1})$$
$$+\frac{10a_i}{12h}(y_i-y_{i-1})+\frac{a_{i+1}}{24h}(y_{i-1}-4y_i+3y_{i+1})$$
$$+\frac{b_{i-1}}{12}y_{i-1}+\frac{10b_i}{12}y_i+\frac{b_{i+1}}{12}y_{i+1}=\frac{(f_{i-1}+10f_i+f_{i+1})}{12}$$

Now introducing the fitting factor $\sigma(\rho)$ in the above scheme

$$\sigma(\rho)\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)+\frac{a_{i-1}}{24h}(-3y_{i-1}+4y_i-y_{i+1})$$
$$+\frac{10a_i}{12h}(y_{i+1}-y_i)+\frac{a_{i+1}}{24h}(y_{i-1}-4y_i+3y_{i+1})+\frac{b_{i-1}}{12}y_{i-1}$$
$$+\frac{10b_i}{12}y_i+\frac{b_{i+1}}{12}y_{i+1}=\frac{(f_{i-1}+10f_i+f_{i+1})}{12} \quad (13)$$

Proceeding as in the left-end boundary layer and using Eq. (11), we get the fitting factor as

$$\sigma=\rho\frac{a(1)}{12}\left(\frac{12(e^{a(1)\rho}-1)}{4\sinh^2(\frac{a(1)\rho}{2})}-1\right) \quad (14)$$

which is a constant fitting factor.

The tridiagonal system of the Eq. (13) is given as follows:

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i, \quad i=l+1,l+2,\ldots,N-1 \quad (15)$$

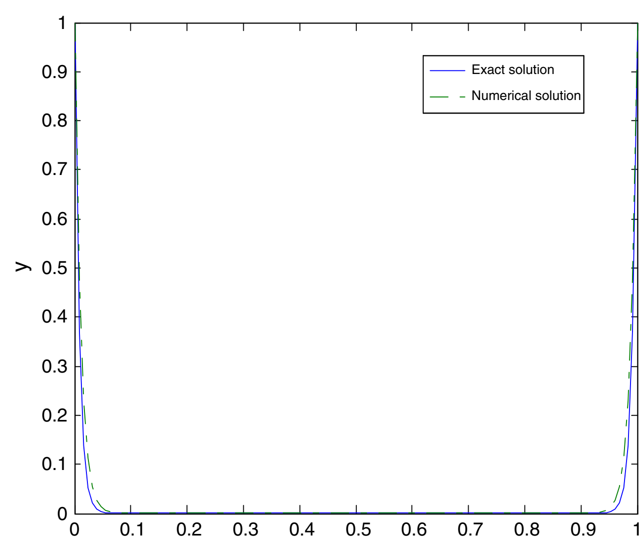

Figure 1 Exact and approximate solutions of Example 1 for $\varepsilon = 2^{-6}$ and $N = 64$ without fitting factor (artificial viscosity).

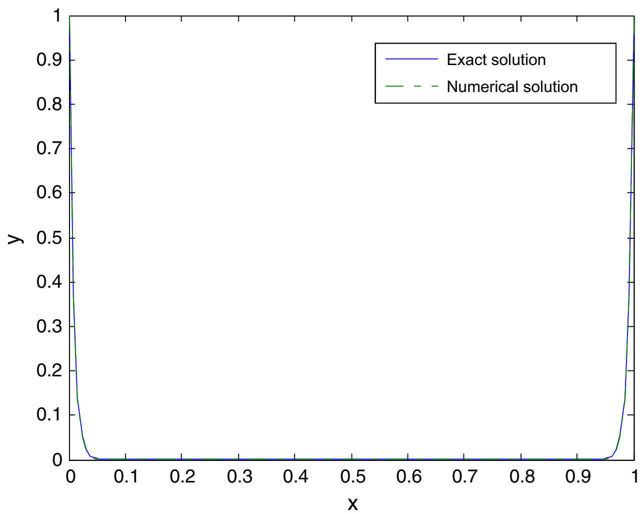

Figure 2 Exact and approximate solutions of Example 1 for $\varepsilon = 2^{-6}$ and $N = 64$ with fitting factor (artificial viscosity).

where

$$E_j = \frac{\varepsilon\sigma}{h^2} - \frac{3a_{i-1}}{24h} + \frac{b_{i-1}}{12} + \frac{a_{i+1}}{24h} - \frac{10a_i}{24h}$$
$$F_j = \frac{2\varepsilon\sigma}{h^2} - \frac{4a_{i-1}}{24h} - \frac{10b_i}{12} + \frac{4a_{i+1}}{24h}$$
$$G_j = \frac{\varepsilon\sigma}{h^2} - \frac{a_{i-1}}{24h} + \frac{b_{i+1}}{12} + \frac{10a_i}{12h} + \frac{3a_{i+1}}{24h}$$
$$H_j = \frac{1}{12}(f_{i-1} + 10f_i + f_{i+1})$$

where σ is given by Eq. (14).

We now have from Eq. (10) in $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, l-1$; and Eq. (15) in $[x_i, x_{i+1}]$ for $i = l+1, l+2, \ldots, N-1$; we get a system of $(N-2)$ equations with $(N+1)$ unknowns. From the given boundary conditions Eq. (2) we get two more equations. We need one more equation to solve the unknowns $y_0, y_1, \ldots, y_N$.

To get this equation, we consider the original differential equation (1) at $x = x_l = 0$.

Since $a(x) = 0$ at $x = x_l = 0$, we get the $\varepsilon y''(x_l) + b(x_l)y = f(x_l)$ (16)

By making use of the central finite difference approximation for the second order derivative in Eq. (16) at x_l, we get

$$[\varepsilon]y_{l-1} - [2\varepsilon - h^2 b_l]y_l + [\varepsilon]y_{l+1} = h^2 f_l \tag{17}$$

With the Eq. (17), we solve the tridiagonal algebraic system Eq. (10), Eq. (15) by using an efficient and stable discrete invariant imbedding algorithm [11].

3. Convergence analysis

Writing the tridiagonal system Eq. (10) in matrix–vector form, we get

$$AY = C \tag{18}$$

in which $A = (m_{ij}), 1 \leqslant i, j \leqslant l-1$ is a tridiagonal matrix of order $l-1$, with

$$m_{ii+1} = \frac{\varepsilon\delta}{h^2} - \frac{a_{i-1}}{24h} + \frac{10a_i}{12h} + \frac{3a_{i+1}}{24h} + \frac{b_{i+1}}{12}$$
$$m_{ii} = \frac{-2\varepsilon\delta}{h^2} + \frac{4a_{i-1}}{24h} - \frac{10a_i}{12h} - \frac{4a_{i+1}}{24h} + \frac{10b_i}{12}$$
$$m_{ii-1} = \frac{\varepsilon\delta}{h^2} - \frac{3a_{i-1}}{24h} + \frac{a_{i+1}}{24h} + \frac{b_{i-1}}{12}$$

and $C = (d_i)$ is a column vector with $d_i = \frac{1}{12}(f_{i-1} + 10f_i + f_{i+1})$ where $i = 1, 2, \ldots, l-1$ with local truncation error

$$T_i(h_i) = h\frac{10}{24}a_i y_i'' + O(h^2) \tag{19}$$

i.e., truncation error in the difference scheme is of $O(h)$.

Writing the tridiagonal system Eq. (15) in matrix–vector form, we get

$$AY = C \tag{20}$$

in which $A = (m_{ij}), l+1 \leqslant i, j \leqslant N-1$ is a tridiagonal matrix of order $N-1$, with

$$m_{ii+1} = \frac{\varepsilon\delta}{h^2} - \frac{a_{i-1}}{24h} + \frac{3a_{i+1}}{24h} + \frac{b_{i+1}}{12}$$
$$m_{ii} = \frac{-2\varepsilon\delta}{h^2} + \frac{4a_{i-1}}{24h} + \frac{10a_i}{12h} - \frac{4a_{i+1}}{24h} + \frac{10b_i}{12}$$
$$m_{ii-1} = \frac{\varepsilon\delta}{h^2} - \frac{3a_{i-1}}{24h} + \frac{a_{i+1}}{24h} + \frac{b_{i-1}}{12} - \frac{10a_i}{12h}$$

and $C = (d_i)$ is a column vector with $d_i = \frac{1}{12}(f_{i-1} + 10f_i + f_{i+1})$, where $i = l+1(1)N-1$ with local truncation error $T_i(h) = -h\frac{10}{24}a_i y_i'' + O(h^2)$ and $Y = (y_0, y_1, y_2, \ldots, _N)^t$.

We also have $A\overline{Y} - T(h) = C$ (21)

where $\overline{Y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N)^t$ denotes the actual solution and $T(h) = (T_0(h_0), T_1(h_1), \ldots, T_N(h_N))^t$ is the local truncation error.

From Eq. (18), Eq. (20) and Eq. (21), we get

$$A(\overline{Y} - Y) = T(h) \tag{22}$$

Thus the error equation is

$$AE = T(h) \tag{23}$$

where $E = \overline{Y} - Y = (e_0, e_1, e_2, \ldots e_N)^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{ij} = \frac{-\varepsilon\delta}{h^2} + \frac{3a_{i-1}}{24h} - \frac{a_{i+1}}{24h} + \frac{b_{i+1}}{12} + \frac{10b_i}{12} \text{ for } i = 1$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = b_i + O(h^2) = B_{i_0}, \text{ for } i = 2, 3, \ldots, N-2$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = \frac{-\varepsilon\delta}{h^2} + \frac{1}{24h}(a_{i-1} - 3a_{i+1}) + \frac{1}{12}(b_{i-1} + 10b_{i+1}), \text{ for } \ i = N-1$$

Since $0 < \varepsilon \ll 1$ and $\delta = o(\varepsilon)$, the matrix A is irreducible and monotone. Then, it follows that A^{-1} exists and its elements are nonnegative.

Hence from Eq. (23), we get

$$E = A^{-1}T(h) \tag{24}$$

and

$$\|E\| \leq \|A^{-1}\| \cdot \|T(h)\| \tag{25}$$

Let $\bar{m}_{ki}$ be the (ki)th element of A^{-1}. Since $\bar{m}_{ki} \geq 0$, from the theory of matrices we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i}\ S_i = 1, \quad k = 1, 2, \ldots, N-1 \tag{26}$$

Table 1 Maximum pointwise errors in Example 1.

	N						
ε	16	32	64	128	256	512	1024
10^{-0}	8.2591(−3)	4.5149(−3)	2.3562(−3)	1.2031(−3)	6.0782(−4)	3.0548(−4)	1.5313(−4)
10^{-1}	3.1815(−2)	1.5007(−2)	7.7688(−3)	3.9601(−3)	2.0040(−3)	1.0085(−3)	5.0598(−4)
10^{-2}	2.1510(−2)	1.0947(−2)	6.8849(−3)	3.9899(−3)	1.7976(−3)	9.0959(−4)	4.6719(−4)
10^{-3}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3159(−3)	7.7335(−4)	4.9079(−4)
10^{-4}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3046(−3)	6.5168(−4)	3.2568(−4)
10^{-5}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3046(−3)	6.5168(−4)	3.2568(−4)
10^{-6}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3046(−3)	6.5168(−4)	3.2568(−4)
10^{-7}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3046(−3)	6.5168(−4)	3.2568(−4)
10^{-8}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3046(−3)	6.5168(−4)	3.2568(−4)
10^{-9}	2.1505(−2)	1.0582(−2)	5.2493(−3)	2.6144(−3)	1.3046(−3)	6.5168(−4)	3.2568(−4)
Results in Natesan et al. [9]							
10^{-0}	0.0079	0.0038	0.0019	0.0009	0.0005	0.0002	0.0001
10^{-1}	0.1354	0.0785	0.0432	0.0229	0.0118	0.0060	0.0030
10^{-2}	0.1753	0.1156	0.0786	0.0487	0.0293	0.0170	0.0096
10^{-3}	0.1792	0.1176	0.0798	0.0494	0.0298	0.0172	0.0097
10^{-4}	0.1796	0.1177	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-5}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-6}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-7}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-8}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-9}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097

Table 2 Maximum pointwise errors in Example 2.

	N						
ε	16	32	64	128	256	512 1024	
10^{-0}	8.2249(−3)	4.5481(−3)	2.3915(−3)	1.2261(−3)	6.2073(−4)	3.1230(−4)	1.5663(−4)
10^{-1}	5.0145(−2)	2.2015(−2)	1.1252(−2)	5.7709(−3)	2.9322(−3)	1.4796(−3)	7.4327(−4)
10^{-2}	6.4595(−2)	3.0857(−2)	1.4431(−2)	6.5769(−3)	2.6641(−3)	1.3391(−3)	6.9204(−4)
10^{-3}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8433(−3)	3.8591(−3)	1.8229(−3)	8.5519(−4)
10^{-4}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8457(−3)	3.9145(−3)	1.9552(−3)	9.7704(−4)
10^{-5}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8457(−3)	3.9145(−3)	1.9552(−3)	9.7708(−4)
10^{-6}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8457(−3)	3.9145(−3)	1.9552(−3)	9.7708(−4)
10^{-7}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8457(−3)	3.9145(−3)	1.9552(−3)	9.7708(−4)
10^{-8}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8457(−3)	3.9145(−3)	1.9552(−3)	9.7708(−4)
10^{-9}	6.4694(−2)	3.1789(−2)	1.5758(−2)	7.8457(−3)	3.9145(−3)	1.9552(−3)	9.7708(−4)
Results in Natesan et al. [9]							
10^{-0}	0.0098	0.0048	0.0023	0.0012	0.0006	0.0003	0.0001
10^{-1}	0.1693	0.1065	0.0609	0.0332	0.0174	0.0089	0.0045
10^{-2}	0.1334	0.1094	0.0832	0.0591	0.0360	0.0221	0.0127
10^{-3}	0.1390	0.1126	0.0848	0.0600	0.0365	0.0224	0.0129
10^{-4}	0.1396	0.1129	0.0849	0.0601	0.0366	0.0225	0.0129
10^{-5}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-6}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-7}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-8}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-9}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129

Therefore

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} \leqslant \frac{1}{\min\limits_{1 \le i \le N-1} S_i} = \frac{1}{B_{i_o}} \leqslant \frac{1}{|B_{i_o}|} \tag{27}$$

for some i_0 between 1 and $N-1$ and $B_{i0} = b_i$We define $\|A^{-1}\| = \max\limits_{1 \leqslant k \leq N-1} \sum_{i=1}^{N-1} |\bar{m}_{ki}|$ and $\|T(h)\| = \max\limits_{1 \leq i \leq N-1} |T_i(h)|$.

From (19), (26) and (24), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{ki} T_i(h), \quad j = 1, 2, 3, \ldots, N-1$$

which implies $\quad e_j \leq \dfrac{kh}{|b_i|}, j = 1(1) \quad N-1 \qquad (28)$

where $k = \frac{10}{24} a_i y_i''$.

Therefore, using Eq. (28), we have $\|E\| = O(h)$i.e., our method reduces to a first order convergent on uniform mesh.

4. Numerical examples

To demonstrate the applicability of proposed method computationally, we consider three examples. These problems have been chosen because they have been widely discussed in the literature.

Example 1. Consider the following singularly perturbed turning point problem

$$\varepsilon y''(x) - 2(2x-1)y'(x) - 4y(x) = 0; x \in (0,1)$$

with $y(0) = 1$ and $y(1) = 1$.

The exact solution of this problem is $y(x) = e^{-2x(1-x)/\varepsilon}$. This problem have two boundary layers, one at $x = 0$ and another at $x = 1$.

The maximum pointwise errors E_ε^N in computed solution are presented in Table 1. The rate of convergence is calculated using double mesh principle [10] given by $p = \log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right)$. The rate of convergence is shown in Table 3. The graphical representation for the solution of this problem is shown in Figs. 1 and 2 without and with fitting factor to show the layer behavior.

Example 2. Consider the singular perturbation problem

$$\varepsilon y''(x) - 2(2x-1)y'(x) - 4y(x) = 4(4x-1); x \in (0,1)$$

with $y(0) = 1$ and $y(1) = 1$.

This problem have two boundary layers at $x = 0$ and at $x = 1$. The exact solution of this problem is not available. To calculate the maximum pointwise errors and rate of convergence, we use double mesh principle [10], $G_\varepsilon^N = \max_{x_i \in D_\varepsilon^N} |Y^N(x_i) - Y^{2N}(x_i)|$ and $G^N = \max_\varepsilon G_\varepsilon^N$ where $Y^N(x_i)$ and $Y^{2N}(x_i)$ denote the numerical solutions obtained using N and $2N$ mesh intervals respectively. Further, we calculate the rate of convergence using $q = \log_2\left(\frac{G_\varepsilon^N}{G_\varepsilon^{2N}}\right)$.

The maximum pointwise errors are presented in Table 2 and rate of convergence is shown in Table 4.

Example 3. Consider the following singular perturbation problem

$$\varepsilon y'' - xy' - y = 0, -1 \le x \le 1$$

with $y(-1) = 1$ and $y(1) = 2$

The exact solution of this problem is not available.

Table 3 Rate of convergence for Example 1.

ε	N 16	32	64	128	256	512
10^{-0}	0.8713	0.9382	0.9697	0.9850	0.9926	0.9963
10^{-1}	1.0841	0.9499	0.9722	0.9827	0.9907	0.9951
10^{-2}	0.9745	0.6690	0.7871	1.1503	0.9828	0.9612
10^{-3}	0.9745	0.6690	0.7871	1.1503	0.9828	0.9612
10^{-4}	0.9745	0.6690	0.7871	1.1503	0.9828	0.9612
10^{-5}	0.9745	0.6690	0.7871	1.1503	0.9828	0.9612

Table 4 Rate of convergence for Example 2.

ε	N 16	32	64	128	256	512
10^{-0}	0.8547	0.9273	0.9638	0.9820	0.9910	0.9955
10^{-1}	1.1876	0.9683	0.9633	0.9768	0.9867	0.9932
10^{-2}	1.0658	1.0964	1.1337	1.3038	0.9923	0.9523
10^{-3}	1.0251	1.0124	1.0066	1.0232	1.0820	1.0919
10^{-4}	1.0251	1.0124	1.0061	1.0031	1.0015	1.0008
10^{-5}	1.0251	1.0124	1.0061	1.0031	1.0015	1.0008

Table 6 Rate of convergence for Example 3.

ε	N 16	32	64	128	256	512
10^{-0}	0.8387	0.9200	0.9608	0.9804	0.9903	0.9952
10^{-1}	1.1858	0.9664	0.9626	0.9764	0.9865	0.9931
10^{-2}	1.0658	1.0965	1.1337	1.3037	0.9923	0.9523
10^{-3}	1.0252	1.0123	1.0066	1.0232	1.0821	1.0918
10^{-4}	1.0252	1.0123	1.0062	1.0031	1.0015	1.0008
10^{-5}	1.0252	1.0123	1.0062	1.0031	1.0015	1.0008

Table 5 Maximum pointwise errors for Example 3.

ε	N 16	32	64	128	256	512	1024
10^{-0}	5.9693(−3)	3.3375(−3)	1.7638(−3)	9.0616(−4)	4.5925(−4)	2.3117(−4)	1.1597(−4)
10^{-1}	3.3476(−2)	1.4715(−2)	7.5304(−3)	3.8639(−3)	1.9637(−3)	9.9108(−4)	4.9791(−4)
10^{-2}	4.3064(−2)	2.0572(−2)	9.6204(−3)	4.3846(−3)	1.7761(−3)	8.9276(−4)	4.6136(−4)
10^{-3}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2289(−3)	2.5727(−3)	1.2152(−3)	5.7013(−4)
10^{-4}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2305(−3)	2.6097(−3)	1.3035(−3)	6.5136(−4)
10^{-5}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2305(−3)	2.6097(−3)	1.3035(−3)	6.5139(−4)
10^{-6}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2305(−3)	2.6097(−3)	1.3035(−3)	6.5139(−4)
10^{-7}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2305(−3)	2.6097(−3)	1.3035(−3)	6.5139(−4)
10^{-8}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2305(−3)	2.6097(−3)	1.3035(−3)	6.5139(−4)
10^{-9}	4.3130(−2)	2.1192(−2)	1.0506(−2)	5.2305(−3)	2.6097(−3)	1.3035(−3)	6.5139(−4)

For this problem, we have two boundary layers one at $x = -1$ and another at $x = 1$.

The maximum pointwise errors are presented in Table 5 and rate of convergence is presented in Table 6.

5. Discussions and conclusion

We have described and demonstrated the applicability of the fitted operator finite difference method for singularly perturbed two-point boundary value problems with dual layers. This method provides an alternative and supplementary technique to the conventional ways of solving singular perturbation problems. We have discussed convergence of the proposed method. To show the efficiency of the method, we have implemented this method on three model examples having dual layer behavior and compared the results with Natesan et al. [9]. It can be observed that the accuracy predicted can always be achieved with very little computational effort with linear convergence. To show the importance of the fitting factor, we have presented the graphical solution of Example 1 with and without fitting factor.

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